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MODELING REVERSE PROBLEMS OF HEAT CONDUCTION WITH MOVING PHASE TRANSITION BOUNDARIES

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An approximate method of solving some reverse problems of nonlinear heat transfer is considered. A procedure is shown for modeling such problems on grid analogs.

We consider a problem of transient heat transfer or diffusion describable by the equations

$$\rho(U)\frac{\partial U}{\partial t}=[\lambda(U)U_x]_x-c(U)U+f(x,\ t),\quad 0<x<\xi(t),\tag{1}$$

$$\bar{\rho}(\bar{U})\frac{\partial \bar{U}}{\partial t}=[\bar{\lambda}(\bar{U})\bar{U}_x]_x-\bar{c}(\bar{U})\bar{U}+\bar{f}(x,\ t),\ \xi(t)<x<l\tag{2}$$

(where ρ, $\bar{\rho}$, λ, $\bar{\lambda}$, c, $\bar{c}$, f, $\bar{f}$ are known functions) with the initial conditions

$$U(x,\ 0)=\varphi(x),\quad 0<x<\xi(0)=\xi_0;\quad \bar{U}(x,\ 0)=\bar{\varphi}(x),\ \xi_0<x<l\tag{3}$$

(including the possible case $\xi_0 = 0$) and the boundary condition

$$[\bar{\mu}(\bar{U})\bar{U}_x+\bar{\nu}(\bar{U})\bar{U}]_{x=l}=-\bar{q}(t,\ \bar{U}(l,\ t)).\tag{4}$$

The law according to which the interphase boundary $\xi(t)$ moves is described by the equation

$$\left\{\gamma(t,\ U,\ \bar{U})\frac{d\xi}{dt}=\bar{\lambda}(\bar{U})\bar{U}_x-\lambda(U)U_x+\Phi(x,\ t,\ U,\ \bar{U})\right\}_{x=\xi(t)},\tag{5}$$

with $\xi(t)$ a monotonically increasing function. At points on the interphase boundary $\xi(t)$ are stipulated the additional constrants

$$U(\xi(t),\ t)=\bar{U}(\xi(t),\ t)=U^0=\text{const}\tag{6}$$

for the Stefan problem or

$$U(\xi(t),\ t)=\bar{U}(\xi(t),\ t),\tag{7}$$

$$\alpha U_x(\xi(t),\ t)=\bar{\alpha}\bar{U}_x(\xi(t),\ t)\tag{8}$$

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(α, $\bar{\alpha}$ are constants) for the Verigin problem. Functions φ, $\bar{\varphi}$, $\bar{q}$, γ, Φ and constants U^0, α, $\bar{\alpha}$ are given.

In addition, there is satisfied one of the two conditions:

A. For either problem (1)-(6) or problem (1)-(5), (7), (8) the location of the interphase boundary $\xi(t)$ is known from an experiment.

B. From an experiment are also known the values of $U(x, t)$ at point $x = a$ ($0 < a < \xi(t)$ for problem (1)-(6), and $0 < a \leqslant \xi(t)$ for problem (1)-(5), (7), (8)).

We now formulate the reverse problem of heat conduction: find the thermal flux $U_x(0, t) = -q(t)$ at the left-hand boundary $x = 0$ and the temperature fields $U(x, t)$, $\bar{U}(x, t)$, and also the location of the interphase boundary if the latter is not known from an experiment.

This problem is an incorrect one, which makes it difficult here to use approximate methods of solution. Known finite-difference schemes approximating such a problem, e.g., are unstable here. When the time step is larger than critical [1], then various regularizing algorithms have to be used for suppressing the instability which builds up [1, 2]. When the time step is larger than critical, then no instability occurs.

Electrical models [3] yield excellent results for a sufficiently coarse grid.

Here a procedure will be proposed for modeling the finite-difference analog of the given problem. It is based on an implicit scheme with trapping the phase front into a node of the grid [4] and then modeling it on an R-grid analog [5]. The selected difference scheme is readily realizable on an R-grid and is absolutely stable for solution of the forward boundary-value problem of heat conduction.

We use the difference scheme $x_i = ih$ ($i = 1, \ldots, N-1$) with $x_k = kh = \xi_0$. The time steps τ_k will be made dependent on h so that each time interval τ_k corresponds to a change of the quantity $\xi(t)$ by h: $\xi_k - \xi_{k-1} = h$. We now rewrite the system of equations (1)-(8) in difference form:

$$\rho_{i,n-1}\frac{U_{in}-U_{i,n-1}}{\tau_n}=\frac{1}{h^2}[\lambda_{i,n-1}(U_{i+1,n}-U_{in})-\lambda_{i-1,n-1}(U_{in}-U_{i-1,n})]-c_{i,n-1}U_{in}+f_{i,n-1},\quad i=1,2,\ldots,k+n-1, \tag{9}$$

$$\bar{\rho}_{i,n-1}\frac{\bar{U}_{in}-\bar{U}_{i,n-1}}{\tau_n}=\frac{1}{h^2}[\bar{\lambda}_{i,n-1}(\bar{U}_{i+1,n}-\bar{U}_{in})-\bar{\lambda}_{i-1,n-1}(\bar{U}_{in}-\bar{U}_{i-1,n})]-\bar{c}_{i,n-1}\bar{U}_{in}+\bar{f}_{i,n-1},\quad i=k+n+1,\ldots,N-1, \tag{10}$$

$$U_{i0}=\varphi_i\,(i=1,\ldots,k),\quad \bar{U}_{i0}=\bar{\varphi}_i\,(i=k,\ldots,N-1), \tag{11}$$

$$\bar{\mu}_{n-1}(\bar{U}_{Nn}-\bar{U}_{N-1,n})+h\bar{\nu}_{n-1}\bar{U}_{Nn}=-h\bar{q}_{n-1}, \tag{12}$$

$$\gamma_{n-1}\frac{\xi_n-\xi_{n-1}}{\tau_n}=\frac{1}{h}[\bar{\lambda}_{k+n,n-1}(\bar{U}_{k+n+1,n}-\bar{U}_{k+n,n})-\lambda_{k+n-1,n-1}(U_{k+n,n}-U_{k+n-1,n})]+\Phi_{k+n,n-1}=\gamma_{n-1}\frac{h}{\tau_n}. \tag{13}$$

At points of the boundary $\xi(t)$ we have

$$U_{k+n,n}=\bar{U}_{k+n,n}=U^0 \tag{14}$$

for the Stefan problem and

$$U_{k+n,n}=\bar{U}_{k+n,n} \tag{15}$$

$$\alpha(U_{k+n,n}-U_{k+n-1,n})=\bar{\alpha}(\bar{U}_{k+n+1,n}-\bar{U}_{k+n,n}) \tag{16}$$

for the Verigin problem.

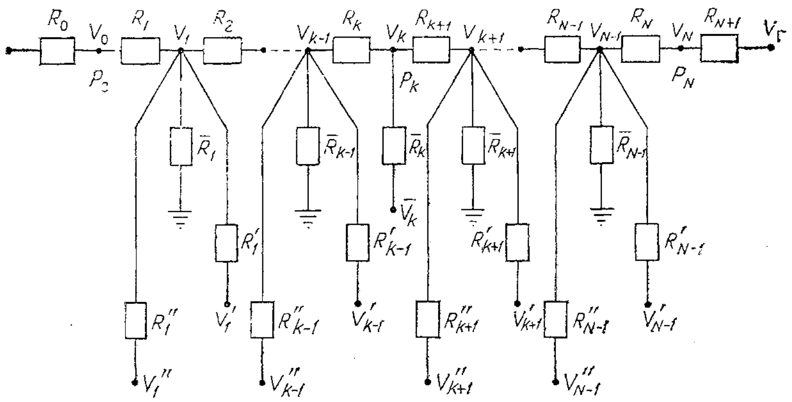

Fig. 1. Electrical model network for systems of equations (9)-(12), (14) and (9)-(12), (15), (16).

This difference scheme gives an error of the order $h + \tau_n$.

Let us then examine the electrical network shown in Fig. 1. Here V_{on}, ..., V_{Nn} denote the voltages at the nodes P_0, ..., P_N, correspondingly. It is well known [5] that with the network parameters selected so as to make

$$R_i = \frac{h^2R}{\lambda_{i,n-1}}, \quad \bar{R}_i = \frac{\tau_n R}{\rho_{i,n-1}}, \quad R'_i = \frac{R}{c_{i,n-1}} \quad (i = 1, \ldots, k+n-1),$$

$$R_i = \frac{h^2R}{\bar{\lambda}_{i,n-1}}, \quad \bar{R}_i = \frac{\tau_n R}{\bar{\rho}_{i,n-1}}, \quad R'_i = \frac{R}{c_{i,n-1}} \quad (i = k+n+1, \ldots, N-1),$$

$$R_{k+n} = \frac{h^2R}{\lambda_{k+n,n-1}}, \quad R_N = \frac{h^2R}{\lambda_{k+n,n-1}}, \quad \bar{R}_{k+n} = 0, \quad R_g = \frac{\bar{\mu}_{n-1}}{h\bar{\nu}_{n-1}} R_N,$$

and with the voltages at the free terminals of the resistance branches stipulated as

$$V_{k+n} = U^0V, \quad V_g = \bar{q}_n R_N \frac{V}{h\bar{\nu}_{n-1}}, \quad V_{i,n-1} = U_{i,n-1}V, \; (V_{i0} = \varphi_i V),$$

where $V''_i = R''f_{i,n-1}V$ (or $V''_i = R''\bar{f}_{i,n-1}V$, respectively), and R and V are, respectively, the resistance scale and the voltage scale, R'' being a sufficiently high resistance, the system of equations for voltages V_{in} at nodes P_i of the electrical network will be identical to the finite-difference system of equations (9)-(12), (14) for the Stefan problem.

With the parameters of the network and the voltages in it such that

$$R_i = \frac{h^2R}{\lambda_{i,n-1}}, \quad \bar{R}_i = \frac{\tau_n R}{\rho_{i,n-1}}, \quad R'_i = \frac{R}{c_{i,n-1}} \quad (i = 1, \ldots, k+n-1),$$

$$R_i = \frac{mh^2R}{\bar{\lambda}_{i,n-1}}, \quad \bar{R}_i = \frac{m\tau_n R}{\bar{\rho}_{i,n-1}}, \quad R'_i = \frac{mR}{\bar{c}_{i,n-1}} \quad (i = k+n+1, \ldots, N-1),$$

$$R_{k+n} = \frac{h^2R}{\lambda_{k+n,n-1}}, \quad R_N = \frac{mh^2R}{\bar{\lambda}_{N,n-1}}, \quad \bar{R}_{k+n} = \infty, \quad R_g = \frac{\mu_{n-1}}{h\bar{\nu}_{n-1}} R_N,$$

$$V_g = \bar{q}_{n-1} R_N \frac{1}{h\bar{\nu}_{n-1}} V, \quad V_{i,n-1} = U_{i,n-1}V \; (V_{i0} = \varphi_i V), \quad V''_i = R''f_{i,n-1}V$$

(or $V''_i = mR''\bar{f}_{i,n-1}V$, respectively), where $m = \bar{\lambda}_{k+n,n-1}\alpha/\lambda_{k+n,n-1}\bar{\alpha}$, the system of equations for voltages V_{in} at nodes P_i of the network will be identical to the finite-difference system of equations (9)-(12), (15), (16) for the Verigin problem.

The reverse problem is modeled on this R-grid according to the following procedure.

When condition A is satisfied, then in each t_n-layer the quantity q_n (or U_{on}) is selected so that condition (13) will be satisfied in node P_{k+n} within an accuracy governed by the accuracy of the approximation and the accuracy of the hardware. One subsequently finds on the model the values U_{in} and $\bar{U}_{jn}$ characterizing the temperature field in both phases. For proceeding to the next time layer, one shifts the point P_{n+k} on the interphase boundary to the next node to the right. The changes to be made in the electrical network in the vicinity of point P_{k+n} are simple and obvious. On the modified grid one finds q_{n+1}, $U_{i,n+1}$, $\bar{U}_{j,n+1}$, etc.

When condition B is satisfied, then along with q_n, U_{in}, and $\bar{U}_{jn}$ one must also find $\xi(t_n)$, i.e., the quantities τ_n. In this case τ_n, $\bar{U}_{jn}$, and U_{in} $(a = x_m < x_i)$ can be determined from the solution to the forward problem

$$\rho_{i,n-1}\frac{U_{in}-U_{i,n-1}}{\tau_n}=\frac{1}{h^2}\,[\lambda_{i,n-1}(U_{i+1,n}-U_{in})-\lambda_{i-1,n-1}(U_{in}-U_{i-1,n})]-c_{i,n-1}U_{in}+f_{i,n-1},\quad m<i<k+n, \tag{17}$$

$$U_{i0}=\varphi_i,\ m\leqslant i<k,\quad U_{mn}=U(a,\ t_n),$$

$$\bar{\rho}_{i,n-1}\frac{\bar{U}_{in}-\bar{U}_{i,n-1}}{\tau_n}=\frac{1}{h^2}\,[\bar{\lambda}_{i,n-1}(\bar{U}_{i+1,n}-\bar{U}_{in})-\bar{\lambda}_{i-1,n-1}(\bar{U}_{in}-\bar{U}_{i-1,n})]-\bar{c}_{i,n-1}\bar{U}_{in}+\bar{f}_{i,n-1},\quad k+n<i<N, \tag{18}$$

$$\bar{U}_{i0}=\bar{\varphi}_i,\quad k\leqslant i<N,\quad \bar{\mu}_{n-1}(\bar{U}_{Nn}-\bar{U}_{N-1,n})+h\bar{\nu}_{n-1}\bar{U}_{Nn}=-h\bar{q}_{n-1}$$

in region $a < x < l$. Equations (17) and (18) must be supplemented with conditions (13)-(14) or (13), (15)-(16).

This second forward problem can be solved by the method of iterations in the t_n-layer according to the scheme

$$\rho_{i,n-1}\frac{U_{in}^{(s)}-U_{i,n-1}}{\tau_n^{(s)}}=\frac{1}{h^2}\,[\lambda_{i,n-1}(U_{i+1,n}^{(s)}-U_{in}^{(s)})-\lambda_{i-1,n-1}(U_{in}^{(s)}-U_{i-1,n}^{(s)})]-c_{i,n-1}U_{in}^{(s)}+f_{i,n-1},\quad m<i<k+n,$$

$$U_{mn}^{(s)}=U(a,\ t_n),$$

$$\bar{\rho}_{i,n-1}\frac{\bar{U}_{in}^{(s)}-\bar{U}_{i,n-1}}{\tau_n^{(s)}}=\frac{1}{h^2}\,[\bar{\lambda}_{i,n-1}(\bar{U}_{i+1,n}^{(s)}-\bar{U}_{in}^{(s)})-\bar{\lambda}_{i-1,n-1}(\bar{U}_{in}^{(s)}-\bar{U}_{i-1,n}^{(s)})]-\bar{c}_{i,n-1}\bar{U}_{in}^{(s)}+\bar{f}_{i,n-1},\quad k+n<i<N,$$

$$U_{k+n,n}^{(s)}=\bar{U}_{k+n,n}^{(s)}, \tag{19}$$

$$\bar{\mu}_{n-1}(\bar{U}_{Nn}^{(s)}-\bar{U}_{N-1,n}^{(s)})+h\bar{\nu}_{n-1}\bar{U}_{Nn}^{(s)}=-h\bar{q}_{n-1},$$

$$\tau_n^{(s+1)}=\frac{1}{p_{n-1}-\bar{q}_{n-1}+\Phi_{n-1}}\Big\{h\gamma_{n-1}+\tau_n^{(s)}\Big[p_{n-1}-\bar{q}_{n-1}+\frac{\lambda_{k+n-1,n-1}}{h}(U_{k+n,n}^{(s)}-U_{k+n-1,n}^{(s)})-\frac{\bar{\lambda}_{k+n,n-1}}{h}(\bar{U}_{k+n+1,n}^{(s)}-\bar{U}_{k+n,n}^{(s)})\Big]\Big\}, \tag{20}$$

where $p_{n-1}=\frac{1}{h}\lambda_m(U_{m+1,n-1}-U_{m,n-1})$ are known from the preceding layer.

Relations (19) and (20) are used for the Stefan problem. For the Verigin problem they must be replaced with the relation

$$U_{k+n,n}^{(s)}=\bar{U}_{k+n,n}^{(s)},\quad \alpha(U_{k+n,n}^{(s)}-U_{k+n-1,n}^{(s)})=\bar{\alpha}(\bar{U}_{k+n+1,n}^{(s)}-\bar{U}_{k+n,n}^{(s)}),$$

$$\tau_n^{(s+1)}=\frac{1}{\bar{p}_{n-1}+\Phi_{n-1}}\Big\{h\gamma_{n-1}+\tau_n^{(s)}\Big[\bar{p}_{n-1}+\frac{1}{h}(\bar{U}_{k+n+1,n}^{(s)}-\bar{U}_{k+n,n}^{(s)})\Big]\Big\},$$

where $\bar{p}_{n-1}=\frac{1}{h}\bar{\lambda}_{k+n-1,n-1}[\bar{U}_{k+n,n-1}-\bar{U}_{k+n-1,n-1}]$.

TABLE 1. Values of Boundary Function U(0, t) for Successive Locations of Phase Front ξ(t)

$U(0, t)$	1,000	1,010	1,010	0,996	1,000	1,002	1,005	1,002
$\xi(t)$	0,4	0,5	0,6	0,7	0,8	0,9	1,0	1,1
$U(0, t)$	0,998	1,001	0,998	0,999	0,999	0,984	0,995	1,017
$\xi(t)$	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9

In the iteration process U_{i0} and $\bar{U}_{i0}$ are given, $U_{i,n-1}$ and $\bar{U}_{i,n-1}$ are known from the preceding layer, and $\tau_n^{(0)}$ can be made equal to τ_{n-1}. The iteration process is terminated upon reaching the given accuracy for τ_n.

It is essential to point out that the solution of system (17)-(18) together with the corresponding iteration process are easily realizable on the same R-grid for solving the reverse problem. In a certain t_n-layer one models first Eqs. (17)-(18) on the right-hand half of the R-grid (starting at point $x = a$) and then, if necessary, the corresponding iteration process so as to find τ_n, U_{in} $(m < i < k+n)$, $\bar{U}_{in}$ $(k+n < i < N)$, whereupon one connects the left-hand side of the R-grid and solves the reverse problem in this layer just as for the case A. Realization of Eqs. (17)-(18) and of the iteration process does not require any additional modifications of the R-grid or any changes in its parameters with regard to the reverse problem.

As an example we show here the results of modeling, according to this procedure, the problem

$$U_t = 2U_{xx}\,(0 < x < \xi(t),\ t > t_0 = 0.181),\quad \bar{U}_t = \bar{U}_{xx}\,(\xi(t) < x < l = 2),$$
$$U(\xi(t),\ t) = \bar{U}(\xi(t),\ t),\ U_x(\xi(t),\ t) = \bar{U}_x(\xi(t),\ t),$$
$$U(x,\ t_0) = \varphi(x) = 1 - 1.3176\,[1 - \Phi(x/2\sqrt{2t_0})]\quad (0 < x < 0.4),$$
$$\bar{U}(x,\ t_0) = \bar{\varphi}(x) = 1.0402\,[1 - \Phi(x/2\sqrt{t_0})]\quad (0{,}4 < x < 2),$$
$$\bar{U}_x\,(2,\ t) = -0.5869/\sqrt{t}\,\exp(-1/t),$$
$$\xi'(t) = \bar{U}_x(\xi(t),\ t) - 2U_x(\xi(t),\ t),\ \xi(t_0) = \xi_0 = 0.4;\ (\xi(t) = 0.94\sqrt{t},\ U(0,\ t) = 1).$$

The data in Table 1 represent the values of U(0, t) found by this method of modeling on an ÉI-12 electrical integrator for successive locations of the boundary ξ(t) according to the scheme (9)-(13), (15), (16) with an h = 0, 1 step. The calculations were made with time steps larger than critical. The high accuracy of the results is attributable to the fact that the calculations were made within the stability range of the difference scheme.

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